

# Band geometry of fractional topological insulators

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Recent numerical simulations of flat band models with interactions which show clear evidence of fractionalized topological phases in the absence of a net magnetic field have generated a great deal of interest. We provide an explanation for these observations by showing that the physics of these systems is the same as that of conventional fractional quantum Hall phases in the lowest Landau level under certain ideal conditions which can be specified in terms of the Berry curvature and the Fubini study metric of the topological band. In particular, we show that when these ideal conditions hold, the density operators projected to the topological band obey the celebrated  $W_\infty$  algebra. Our approach provides a quantitative way of testing the suitability of topological bands for hosting fractionalized phases.

The advent of topological insulators which are band insulators with topologically non-trivial bands, has generated a great deal of recent interest in topological phases [1–3]. The Landau levels whose filling gives rise to the integer quantum Hall effect [4] can also be regarded as topologically non-trivial bands. While the integer quantum Hall effect has so far only been observed in the presence of large magnetic fields, a quantized Hall conductance can also arise in the presence of a periodic potential, where it can be related to a topological invariant associated with the bands of Bloch wavefunctions [5]. In the absence of any time-reversal symmetry breaking, this invariant has to be zero. However, there exist tight-binding models which explicitly demonstrate that a quantized Hall conductance is possible in a net zero external magnetic field, which albeit break time-reversal symmetry [6].

In the presence of interactions, electrons in fractionally filled Landau levels can form a liquid-like phase with a quantized Hall conductance and a gap to all bulk excitations [7]. This is a topological phase with a non-trivial ground state degeneracy on a torus and excitations with fractional charge and, depending on the filling fraction, fractional or possibly even non-abelian statistics. The question of whether similar phenomena can occur in a band insulator model has recently been addressed in a series of numerical works by many groups [8–12], which provide clear evidence for the existence of gapped phases possessing many of the signatures of the proposed ground states for fractional quantum Hall states.

We provide a rationalization for these surprising developments based on an approach introduced in Ref. 13. We show that under certain ideal conditions which will be specified in detail below, the projected density operators obey a closed algebra which has the same form as the celebrated  $W_\infty$  algebra of the lowest Landau level projected density operators [14–17].

The single particle states of Chern bands are very different from Landau level wavefunctions, so it is not a priori clear why partially filled Chern bands can display an analog of the fractional quantum Hall effect (FQHE). One set of rationalizations for these numerical results has

been based on trial wavefunctions constructed either by mapping lowest Landau level wavefunctions in the Landau gauge to Wannier functions [18] or by partonization [19, 20]. The spectacular success of model wavefunctions such as Laughlin’s famous wavefunction in the theory of the FQHE [21] makes such an approach very attractive, but the lack of an appealing analytic form of the Chern band wavefunctions and their poor overlap with exact ground-state wavefunctions in small systems is in sharp contrast to the model FQHE wavefunctions. This has motivated the search for other explanations.

The essential requirements for the formation of fractional Chern insulators were assumed, in the early numerical work, to be some of the characteristic energetic features of the FQHE. These are: 1. a nearly flat band with a non-trivial topological invariant, and 2. short range interactions whose energy scale is much larger than the band width of the non-trivial band, but much smaller than the band gap. Under these circumstances, it is reasonable to project the interactions to the topological band, as is usually carried out in the theory of the FQHE. The assumption is that the low energy spectrum consists of states whose admixture with components from the other bands can be neglected. Even, with this assumption, however, there are a large number of parameters in the fractional Chern insulator problem. First, there is the freedom in choosing the lattice itself, which breaks both continuous translational and rotational symmetry, but may have certain discrete point and space group symmetries. One may also vary the parameters that determine the detailed form of the interaction and finally, one can also change the various tight binding parameters of the single particle Hamiltonian.

In numerical experiments to date, only a limited portion of this large phase space of possible fractional Chern insulators has been explored. Already, it is clear that there is a great variation in the stability of fractional Chern insulator states even when the basic energetic criteria listed in the previous paragraph are met. One of the aims of this paper is to identify other criteria which affect the stability of fractional Chern insulator states. Our approach is based on studying the commutation re-

lations of projected density operators, a direction which has yielded some success in analyzing fractional Chern insulators [13, 22–24].

Before delving into the role of interactions, we describe the basic framework of (nearly) flat band topological insulators. Our starting point is a tightbinding model which has topologically non-trivial bands, a famous example of which is the Haldane model on a honeycomb lattice [6]. By varying the tightbinding parameters, one can flatten the energy bands without altering the topology of the band structure [25, 26]. The Hamiltonian of an  $N$  band insulator can be written as  $\sum_{\mathbf{k}, p, q} |\mathbf{k}, p\rangle \langle h(\mathbf{k}) \rangle_{pq} \langle \mathbf{k}, q|$ , where the sum over crystal momenta is restricted to the first Brillouin zone (BZ). (Here and for the remainder of the proposal, we will adopt the convention that repeated indices are *not* implicitly summed over. When required, we will indicate a summation explicitly.)

The states  $|\mathbf{k}, p\rangle$  are the Fourier transforms of the localized tightbinding orbital states:

$$|\mathbf{k}, p\rangle = \sum_{\mathbf{R}_n} e^{i\mathbf{k} \cdot (\mathbf{e}_p + \mathbf{R}_n)} |\mathbf{R}_n, p\rangle,$$

where  $\mathbf{e}_p + \mathbf{R}_n$  denotes the position of the (localized)  $p^{\text{th}}$  orbital,  $|\mathbf{R}_n, p\rangle$ , in the  $n^{\text{th}}$  unit cell situated at the lattice vector  $\mathbf{R}_n$ . The matrix  $h(\mathbf{k})$  can be diagonalized through an appropriate unitary transformation and the Hamiltonian written in the form  $H_K = \sum_{\gamma, \mathbf{k}} E_\gamma(\mathbf{k}) |\mathbf{k}, \gamma\rangle \langle \mathbf{k}, \gamma|$  where  $|\mathbf{k}, \gamma\rangle = \sum_p u_p^\gamma(\mathbf{k}) |\mathbf{k}, p\rangle$  and  $(u_j^\gamma(\mathbf{k}))$  is a normalized eigenstate of  $h(\mathbf{k})$  with eigenvalue  $E_\gamma(\mathbf{k})$ .

We will use the label  $\alpha$  for the topological band of interest. The Berry curvature,  $B_\alpha(\mathbf{k})$  of the band is defined as:

$$B_\alpha(\mathbf{k}) = -i \sum_p \left( \frac{\partial u_p^{\alpha*}}{\partial k_x} \frac{\partial u_p^\alpha}{\partial k_y} - \frac{\partial u_p^{\alpha*}}{\partial k_y} \frac{\partial u_p^\alpha}{\partial k_x} \right) \quad (1)$$

and its integral over the Brillouin zone is

$$\int_{BZ} dk_x dk_y B_\alpha(\mathbf{k}) = 2\pi C_\alpha \quad (2)$$

where  $C_\alpha$  is the Chern number of the band,  $\alpha$ . For a topological (Chern) band,  $C_\alpha$  is non-zero and without loss of generality, we can take  $C_\alpha$  to be a positive integer.

We now consider the role of interactions by adding a term  $U_{\text{int}}$  to the Hamiltonian. The interactions we consider are generally (but not always) density-density interactions of the form  $U_{\text{int}} = \sum_{i,j} u(\mathbf{r}_i - \mathbf{r}_j)$ . In the limit of a large band gap, one can safely neglect the mixing between the Chern band and the unfilled bands. If the bandwidth is small compared to the scale of the interactions,  $E_\alpha(k)$  may be treated as constant and may be set to zero by a simple regularization. With this approximation, the low energy effective Hamiltonian including interactions has the form:  $H_{\text{eff}} = \bar{U}_{\text{int}}$ , where  $\bar{U}_{\text{int}}$  is the interaction projected to the Chern band.

One encounters a similar Hamiltonian in the treatment of interactions in the lowest Landau level in a large magnetic field. In that case, the effective Hamiltonian of a clean system obtained by projecting density-density interactions to the lowest Landau level has the form

$$H_{LLL} = \frac{1}{2} \sum_{\mathbf{q}} V(\mathbf{q}) e^{-q^2 \ell_B^2/4} \bar{\rho}_{\mathbf{q}} \bar{\rho}_{-\mathbf{q}} \quad (3)$$

where  $\bar{\rho}_{\mathbf{q}}$  differs from the projected density,  $\mathcal{P} \rho_{\mathbf{q}} \mathcal{P}$  by a  $q$ -dependent constant,  $\mathcal{P} \rho_{\mathbf{q}} \mathcal{P} = e^{-q^2 \ell_B^2/4} e^{i\mathbf{q} \cdot \mathbf{R}} \equiv e^{-q^2 \ell_B^2/2} \bar{\rho}_{\mathbf{q}}$ .

In the LLL problem, the projected density operators,  $\bar{\rho}_{\mathbf{q}}$  obey the  $W_\infty$  algebra, first identified by Girvin, MacDonald and Platzman (GMP) [14–17]:

$$[\bar{\rho}_{\mathbf{q}_1}, \bar{\rho}_{\mathbf{q}_2}] = 2i \sin \left( \frac{\mathbf{q}_1 \wedge \mathbf{q}_2 \ell_B^2}{2} \right) \bar{\rho}_{\mathbf{q}_1 + \mathbf{q}_2} \quad (4)$$

where  $\mathbf{q}_1 \wedge \mathbf{q}_2 \equiv \hat{\mathbf{z}} \cdot (\mathbf{q}_1 \times \mathbf{q}_2)$ . This algebra is the quantum version of the algebra of area-preserving diffeomorphisms on the plane and can also be interpreted as that of magnetic translations in a uniform field [27]. Together, the density algebra of Eq. (4) and the effective Hamiltonian of Eq. (3) capture the non-trivial dynamics that arise from projection to the lowest Landau level.

Since the fractionally filled Chern insulator has an effective Hamiltonian which has the same form as that of the FQHE, if the projected density operators of the fractional Chern insulator also obey the same algebra, the same low energy physics would ensue, explaining the existence of topological phases in flattened Chern bands.

Let us therefore examine the projected density operators of the Chern band, following a strategy outlined in Ref. 13. Let  $P_\alpha = \sum_{\mathbf{k}} |\mathbf{k}, \alpha\rangle \langle \mathbf{k}, \alpha|$  be the operator that projects to the Chern band. A Taylor expansion of the projected density operator,  $\bar{\rho}_{\mathbf{q}} = P_\alpha \rho_{\mathbf{q}} P_\alpha$  keeping only terms of order  $q^2$  yields:

$$\bar{\rho}_{\mathbf{q}} = P_\alpha + iP_\alpha \mathbf{q} \cdot \mathbf{r} P_\alpha - \frac{1}{2} P_\alpha (\mathbf{q} \cdot \mathbf{r})^2 P_\alpha \quad (5)$$

From this expression, it follows that, provided the Berry curvature,  $B_\alpha(\mathbf{k})$  is uniform in momentum space, i.e., provided the fluctuations in the Berry curvature can be neglected, up to order  $q^2$ , the following relation holds:

$$[\bar{\rho}_{\mathbf{q}_1}, \bar{\rho}_{\mathbf{q}_2}] = i(\mathbf{q}_1 \wedge \mathbf{q}_2) \bar{B}_\alpha P_\alpha \quad (6)$$

where  $\bar{B}_\alpha = 2\pi C_\alpha / A_{BZ}$  is the average Berry curvature and  $A_{BZ}$  is the area of the Brillouin zone. One may then assert [13] (that to order  $q^2$ ), this has the same form as the  $W_\infty$  algebra of the LLL projected density operators with  $\sqrt{\bar{B}_\alpha}$  playing the role of the magnetic length,  $l_B$ .

Most band structures do not have a uniform Berry curvature, and thus the relation holds only approximately, even to order  $q^2$ . One can however make a virtue of what

seems like a failing by arguing that the degree of deviation from a uniform Berry curvature provides a way to predict quantify measure how good a host a particular band structure is for hosting FQHE-like phases, an expectation that has been confirmed by numerics [28].

It is natural to consider higher order terms in  $q$  in  $[\bar{\rho}_{\mathbf{q}_1}, \bar{\rho}_{\mathbf{q}_2}]$ . Keeping terms of order  $q^3$ , we find, after a little algebra, that

$$[\bar{\rho}_{\mathbf{q}_1}, \bar{\rho}_{\mathbf{q}_2}] = (i)\bar{B}_\alpha(\mathbf{q}_1 \wedge \mathbf{q}_2)P_\alpha(1 + iP_\alpha(\mathbf{q}_1 + \mathbf{q}_2) \cdot \mathbf{r}P_\alpha) - \frac{i}{2} \sum_{a,b,c} \left( \frac{q_{1a}q_{2b}q_{2c}}{2} [P_\alpha r_a P_\alpha, P_\alpha(r_b Q_\alpha r_c + r_c Q_\alpha r_b)P_\alpha] + \frac{q_{1a}q_{1b}q_{2c}}{2} [P_\alpha(r_a Q_\alpha r_b + r_b Q_\alpha r_a)P_\alpha, P_\alpha r_c P_\alpha] \right)$$

where  $Q_\alpha = I - P_\alpha$  and  $I$  is the identity operator. The commutators  $[P_\alpha r_a P_\alpha, P_\alpha(r_b Q_\alpha r_c + r_c Q_\alpha r_b)P_\alpha]$  and  $[P_\alpha(r_a Q_\alpha r_b + r_b Q_\alpha r_a)P_\alpha, P_\alpha r_c P_\alpha]$  vanish if and only if the Fubini-Study (FS) metric tensor,  $g^\alpha(\mathbf{k})$  is a constant in the Brillouin zone. The Fubini-Study metric is a rank two symmetric tensor,  $g^\alpha(\mathbf{k})$  with components [29–32]

$$g_{ab}^\alpha(\mathbf{k}) = \frac{1}{2} \sum_p \left[ \left( \frac{\partial u_p^{\alpha*}}{\partial k_a} \frac{\partial u_p^\alpha}{\partial k_b} + \frac{\partial u_p^{\alpha*}}{\partial k_b} \frac{\partial u_p^\alpha}{\partial k_a} \right) - \sum_q \left( \frac{\partial u_p^{\alpha*}}{\partial k_b} u_p^\alpha u_q^{\alpha*} \frac{\partial u_q^\alpha}{\partial k_a} + \frac{\partial u_p^{\alpha*}}{\partial k_a} u_p^\alpha u_q^{\alpha*} \frac{\partial u_q^\alpha}{\partial k_b} \right) \right]$$

If the metric tensor is constant in the Brillouin zone, then to order  $q^3$ , the Chern band projected densities satisfy the  $W_\infty$  algebra of projected LLL densities. This leads us to identify the uniformity of the metric tensor in momentum space as an additional “metric” for identifying “good” band structures from the point of view of hosting interacting topological phases. Of course, other conditions such as a suitable short ranged interaction, and a proper hierarchy of energy scales are no less important.

We will see that when the band structure satisfies one additional constraint, the Chern band projected densities satisfy the  $W_\infty$  algebra of projected LLL densities at all orders in  $q$ . If one could completely ignore the Fubini-Study metric tensor, i.e., set it to zero, then with the assumption of a constant Berry curvature, the algebra of projected density operators would simply be the Heisenberg algebra and would close at all wavelengths. However, the FS metric cannot vanish as the non-trivial topology of the band structure of a Chern band places some constraints on the form of the FS metric tensor. The trace of the FS metric tensor at a given point in  $k$  space, which we denote by  $tr(g^\alpha(\mathbf{k}))$  can be expressed as:

$$tr(g^\alpha(\mathbf{k})) = \langle \mathbf{k}, \alpha | (x + iy)(I - P_\alpha)(x - iy) | \mathbf{k}, \alpha \rangle + i \langle \mathbf{k}, \alpha | (x(I - P_\alpha)y - y(I - P_\alpha)x) | \mathbf{k}, \alpha \rangle \quad (7)$$

The positive definiteness of the operators,  $A^\dagger A$  and  $C^\dagger C$  where  $A = (I - P_\alpha)(x + iy)P_\alpha$  and  $C = (I - P_\alpha)(x - iy)P_\alpha$  implies that

$$\langle \mathbf{k}, \alpha | AA^\dagger | \mathbf{k}, \alpha \rangle \geq 0, \quad \langle \mathbf{k}, \alpha | CC^\dagger | \mathbf{k}, \alpha \rangle \geq 0 \quad (8)$$

From these inequalities and Eq. (7), it follows that

$$tr(g^\alpha(\mathbf{k})) \geq |B_\alpha(\mathbf{k})| \quad (9)$$

Thus the magnitude of the Berry curvature places a lower bound on the trace of the Fubini-Study metric.

One may define the transformed operators,  $x'_a = \sum_b t_{ab} x_b$ , corresponding to rotations and scale transformations. Here  $t$  is an invertible matrix and  $x_1 = x, x_2 = y$ .

We may also define a corresponding transformed FS metric,  $g^{\alpha'}(\mathbf{k})$ :

$$g_{ab}^{\alpha'}(\mathbf{k}) = \frac{1}{2} \langle \mathbf{k}, \alpha | (x'_a(I - P_\alpha)x'_b + x'_b(I - P_\alpha)x'_a) | \mathbf{k}, \alpha \rangle$$

Thus,  $g_{ab}^{\alpha'}(\mathbf{k}) = t_{ac} g_{cd}^\alpha(\mathbf{k}) t_{bd}$ . Similarly, the transformed Berry curvature is

$$B'_\alpha(\mathbf{k}) = \sum_{a,b} \epsilon_{ab} \langle \mathbf{k}, \alpha | x'_a P_\alpha x'_b | \mathbf{k}', \alpha \rangle$$

If one chooses  $t$  such that it corresponds to a unimodular coordinate transformation with  $\det(t) = 1$ , the Berry curvature is left unchanged, i.e.,  $B'_\alpha(\mathbf{k}) = B_\alpha(\mathbf{k})$ . The inequality (9) then also applies to the transformed FS metric and the Berry curvatures. Thus,

$$tr(g^{\alpha'}(\mathbf{k})) \geq |B_\alpha(\mathbf{k})| \quad (10)$$

One can always find a unimodular transformation such that the transformed metric at any given point,  $\mathbf{k}_0$  is a diagonal matrix. Since the determinant of the FS metric is preserved through such a transformation, the transformed metric may be written as

$$g^{\alpha'}(\mathbf{k}_0) = \begin{pmatrix} \sqrt{\det(g^\alpha(\mathbf{k}_0))} & 0 \\ 0 & \sqrt{\det(g^\alpha(\mathbf{k}_0))} \end{pmatrix} \quad (11)$$

Applying the inequality (10) to the transformed metric of Eq. (11), we find that  $2\sqrt{\det(g^\alpha(\mathbf{k}_0))} \geq |B_\alpha(\mathbf{k}_0)|$ . Since, the point  $\mathbf{k}_0$  is arbitrary, we conclude that

$$\det(g^\alpha(\mathbf{k})) \geq \frac{|B_\alpha(\mathbf{k})|^2}{4} \quad (12)$$

for any  $\mathbf{k}$  in the BZ. Further, since

$$\int_{BZ} dk_x dk_y (B_\alpha(\mathbf{k}))^2 \geq A_{BZ} \bar{B}_\alpha^2, \quad (13)$$

it follows that

$$\int_{BZ} dk_x dk_y \det(g^\alpha(\mathbf{k})) \geq \frac{A_{BZ} \bar{B}_\alpha^2}{4} = \frac{\pi^2 C_\alpha^2}{A_{BZ}} \quad (14)$$

Thus the integral of the determinant of the FS metric is bounded from below by a number which is proportional to the square of the topological invariant of the band.

Consider now, the case when the inequality (14) is saturated and the FS metric is uniform in the BZ. The inequality (14) is saturated when  $\det(g^\alpha(\mathbf{k})) = \frac{|B_\alpha(\mathbf{k})|^2}{4}$  at all points,  $\mathbf{k}$  in the BZ, and when in addition, the Berry curvature is uniform in the BZ. From the constancy of the FS metric and the saturation of inequality (12), it follows that there is some matrix  $t'$  such that

$$t'g^\alpha(\mathbf{k})(t')^T = \begin{pmatrix} \frac{\bar{B}_\alpha}{2} & 0 \\ 0 & \frac{\bar{B}_\alpha}{2} \end{pmatrix}, \quad (15)$$

where we have assumed without loss of generality that  $\bar{B}_\alpha > 0$ . If  $x', y'$  are the corresponding transformed position operators, from Eq. 7 and the above conditions, it follows that

$$\sum_{\mathbf{k}, \gamma} \langle \mathbf{k}, \gamma | P_\alpha(x' + iy') Q_\alpha(x' - iy') P_\alpha | \mathbf{k}, \gamma \rangle = 0.$$

This implies that the trace of  $DD^\dagger$  where  $D = Q_\alpha(x' - iy')P_\alpha$  is zero and since  $DD^\dagger$  is a positive definite matrix, we may conclude that  $Q_\alpha(x' - iy')P_\alpha = 0$ .

Let  $q'_a = \sum_b t_{ab}^{-1} q_b$ . Writing the density operator  $\rho_{\mathbf{q}}$  as  $e^{i\mathbf{q} \cdot \mathbf{r}} = e^{i\mathbf{q}' \cdot \mathbf{r}'} = e^{\frac{i}{2} \{ (q'_x + iq'_y)(x' - iy') + (q'_x - iq'_y)(x' + iy') \}}$ , it is easy to verify that the density operators satisfy a generalized metric-dependent version of the  $W_\infty$  algebra:

$$[\bar{\rho}_{\mathbf{q}_1}, \bar{\rho}_{\mathbf{q}_2}] = 2i \sin\left(\frac{\mathbf{q}_1 \wedge \mathbf{q}_2 \bar{B}_\alpha}{2}\right) e^{(q_1)_l g_{lm}^\alpha (q_2)_m} \bar{\rho}_{\mathbf{q}_1 + \mathbf{q}_2}$$

In summary, if  $\int_{BZ} dk_x dk_y \det(g^\alpha(\mathbf{k})) = \frac{\pi^2 C_\alpha^2}{A_{BZ}}$  and if the Fubini-Study metric is uniform in the BZ, the density operators satisfy a closed algebra which is a generalization of the usual  $W_\infty$  algebra. We note a couple of interesting features. Firstly, the Berry curvature and the FS metric both appear in this form of the  $W_\infty$  algebra. Thus the algebra also applies to bands, which have a higher Chern number and which therefore differ fundamentally from Landau levels which have Chern number 1. Secondly, we observe that the conditions under which we get a closed algebra of the projected density operators can be stated purely in terms of the FS metric.

For a system where the ideal conditions under which this algebra is obtained do not hold, the degree of deviation from these conditions provides a new parameter (or a set of parameters, depending on how one chooses to quantify the deviation) to predict how favorable a Chern band is for hosting FQHE-like physics. Conversely, if one finds fractional topological phases in systems where the deviations from these conditions is considerable, one could argue that the physics of those systems is new and different from the conventional fractional quantum Hall effect.

The effects of disorder also enter the Hamiltonian through terms that involve the projected density operator. This suggests that the effects of disorder in the Chern band are likely to be the same as in the LLL when the conditions stated above for the FS metric are satisfied.

Let us briefly discuss other band structures where fractional phases may arise [33–35]. One primary example are topological bands with time-reversal symmetry. Consider, for instance, the case of  $Z_2$  insulators with a pair of time-reversed paired flat bands. The projection operator,  $P$  to the topological bands can then always be written as a sum  $P = P_1 + P_2$  where  $P_1$  and  $P_2$  are a pair of projectors related by time-reversal symmetry, which have Chern numbers associated with them that are equal in magnitude and opposite in sign [36, 37]. There may be circumstances where the interactions between electrons with different indices can be neglected either due to the nature of the physical interactions or due to the formation of fractionalized states where the role of such interactions are minimized. Then the relevant projected density operators are  $P_i \rho_{\mathbf{q}} P_i$  ( $i = 1, 2$ ) and the conditions under which these form a closed algebra are the uniformity of the FS metric associated with each projection operator and the saturation of inequality (14) for the same.

Ref. [13] and the current work highlight the important role of geometric features of bands in fractional topological insulators. One is tempted to even go so far as to suggest that the term “fractional topological insulators” should be replaced by “fractional geometric insulators”.

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